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PRODUCTION FUNCTIONS IN FISHERY RESEARCH

K. ALAGARAJA

Central Marine Fisheries Research Institute, Cochin-682 031, India

ABSTRACT

Assessment of aquaproduction assumes all the more importance, at present, as there is global awareness to add more to food supply from water resources by husbanding wild fishery resources. This assessment demands the determination of carrying capacity of water bodies both qualitatively and quantitatively. Here in this paper quantitative approach is touched.

Models developed on the basis of some assumptions on growth, mortality etc. are available. Ricker (1946) and Allen (1950) developed a model to quantify the production by $P = GB$ where G is the instantaneous growth rate and B is the average biomass during the period of reference. Similarly Gulin and Rudenko (1973) also gave a model giving production by $P_t = a_t N_t (1-e^{-zt})/zt$. Similar models can be evolved with varying assumptions.

To find out the differential potentialities of water bodies, error estimates for production functions are required for comparing production of different water bodies and to see whether the differences if any are due to any assignable causes or not. Many models suffer for want of such methods to estimate errors associated with the production functions. Method to estimate error for the model developed by Gulin and Rudenko (1973) is not available. Chapman (1971), however, has tried to find out an error estimate for Ricker's model. In that it is assumed that correlation between G_t and B_t is negative and ommitted.

In this paper various production functions are evaluated and derived complete expressions for their variances leading to their error estimates. The conjecture posed by Chapman (1971) about the negativity and negligibility of the correlation between $\bar{\hat{B}}_t$ and $\hat{G}_t$ is studied and proved that correlation $\hat{B}_t$ and $\hat{G}_t$ is negative whereas that between average $\hat{B}_t$ namely $\bar{\hat{B}}_t$ and $\hat{G}_t$ nothing could be stated. And also this part of variance, namely the covariance between $\hat{G}_t$ and $\bar{\hat{B}}_t$ is accounted for in this paper as this may not be negligible thus giving full expression to the variance function for Ricker's model.

Among different types of models, one based on liner relationship on numbers over time and growth over time is suggested for its simplicity, theoretical soundness and practical applicability. An example is considered and estimates compared along with their variance estimates.

INTRODUCTION

CULTURE practices in confined waters have recently taken great strides in developing as well as developed countries. To determine carrying capacity of a water body it is necessary to evaluate its production. In general, production is not synonymous with yield. To determine production, periodical observations on number and average weight of population are required. Since feeding schedules depend on number of animals and their average weight, such periodical samplings are of much help. Apart from this, difference between production and yield may throw light on factors such as mortality that are responsible for the difference and that may suggest the ways to improve yield. Moreover periodical sampling leads to estimates of vital rates and study of growth under different feeding schedules.

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Following Ricker (1971) production can be defined as the increase in biomass in a given time including the growth of those which die or which are caught during this interval. On the basis of this, Ricker (1946) and Allen (1950) have found the production function

$$P = G\bar{B} \quad .. \quad .. \quad .. \qquad (1)$$

where P is the production, G the instantaneous growth rate in weight and $\bar{B}$ the average biomass during the period of reference. It is assumed here that G does not vary during this period and that growth in weight follows exponential law with time viz :

$W = W_o\, e^{Gt}$ whereas change in numbers may follow any law. Beverton and Holt (1957) have evaluated production function on the assumption that growth in weight follows von Bertalanffy's model and change in numbers follows exponential law with time resulting in

$$P = RW_\infty \sum_{n=0}^{3} \Omega_n\, e^{-nK(t_\rho - t_0)} \frac{1 - e^{-(F+M+nK)(t_\lambda - t_\rho)}}{F+M+nK} \qquad (2)$$

Recently Gulin and Rudenko (1973) have estimated production of lake Demenets by using

$$P_t = a_t\, N_t\, (1 - e^{-Z_t})/Z_t \qquad (3)$$

assuming that growth in weight is linear and change in numbers is exponential with time where

$$N_t\,(\tau) = N_t\, e^{-Z_t \tau}$$

and

$$W_t\,(\tau) = W_t + a_t\, \tau$$

where t is the age of fish, τ any instant in (0, 1). Here the instantaneous mortality rate Z_t etc. is age specific.

Chapman (1971) has given variance function for (1). However, explicit variance function for (3) is not available. Even in case of (1) Chapman (1971) has not given the complete expression for variance function. In this case the contribution from covariance of $\hat{G}_t$ and $\hat{\bar{B}}_t$ is omitted with a remark that correlation between $\hat{G}_t$ and $\hat{B}_t$ as expected, may be negative. The variance function thus arrived at by him is

$$V(\hat{P}_t) = G_t^2\, V(\hat{\bar{B}}_t) + \bar{B}_t^2\, V(\hat{G}_t)$$

To estimate $V(\hat{G}_t)$ he also suggests to take few subsamples leading to few estimates of G_t and from these estimates he gets $\hat{V}(\hat{G}_t) =$

$$\left[\Sigma\, G_t^2 - \frac{(\Sigma\, G_t)^2}{r}\right] \Big/ (r-1)$$

where 'r' is the number of subsamples.

In this paper we shall derive complete expressions for variance functions of different production functions. We shall prove that correlation between $\hat{B}_t$ and $\hat{G}_t$ is negative whereas nothing can be said with certainty about the sign of correlation between B_t and G_t. We shall also take an estimate of $V(G_t)$ from the large sample theory avoiding subsampling approach suggested by Chapman (1971). Finally we shall consider an example taken from Chapman (1971) and see how estimates and their variances obtained from different production functions compare.

PRODUCTION FUNCTIONS

Starting from the definition, in the usual notation we have $dP_t = N_t dW_t$. Assuming without loss of generality that the entire period is divided into unit segments each segment representing a month or a season or a year as the case may be, we may write.

$$P_t = \int_0^1 N_t(T)\, dW_t(T);\ (0 < T < 1) \tag{4}$$

where P_t is the production in $(t, t+1)$ segment. Basing on (4) we shall get different production functions under different assumptions and note down those which are already available in the literature.

Assumption I

$$N_t(T) = N_t e^{-Z_t T}$$

and

$$\bar{W}_t(T) = \bar{W}_t e^{G_t T}$$

then

$$P_{1t} = \frac{N_t \bar{W}_t G_t}{(G_t - Z_t)} [e^{(G_t - Z_t)} - 1] \tag{5}$$

when $G_t > Z_t$

$$= \frac{N_t \bar{W}_t G_t}{(Z_t - G_t)} [e^{(Z_t - G_t)} - 1] \tag{6}$$

when $Z_t > G_t$

$$= N_t W_t G_t \quad \text{when } Z_t = G_t \tag{7}$$

It may be noted that whenever $|G_t - Z_t|$ is sufficiently small such that $N_t \bar{W}_t G_t |G_t - Z_t|$ becomes negligible then (5) and (6) lead to (7). Forms (5) and (6) have been dealt with by Ricker 1971 and Allen 1950 and considered by Chapman 1971.

Assumption II

$$\bar{W}_t(T) = \bar{W}_t e^{G_t T}$$

and no assumption be made on N_t. Then

$$P_{2t} = \int_0^1 G_t N_t(T) \bar{W}_t(T)\, dT$$

$$= G_t \int_0^1 B_t(T)\, dT$$

$$= G_t \bar{B}_t \tag{8}$$

where $B_t(T)$ is the biomass of fish of age 't' at T th instant and $\bar{B}_t$ is the average biomass of fish of age 't' in $(t, t+1)$ segment. (8) has been derived by both Ricker (1971) and Allen (1950).

Assumption III

$$N_t(T) = N_t e^{-Z_t T}$$

and no assumption be made on W_t, then

$$P_{3t} = \int_0^1 N_t(T)\, d\bar{W}_t(T)$$

$$= [N_t(T) \bar{W}_t(T)]_0^1 - \int_0^1 \bar{W}_t(T)\, dN_t(T)$$

$$= N_{t+1} \bar{W}_{t+1} - N_t \bar{W}_t + Z_t \int_0^1 \bar{W}_t(T) N_t(T)\, dT$$

$$= B_{t+1} - B_t + Z_t \bar{B}_t \tag{9}$$

Assumption IV

In any curve, when sufficiently segmented, each segment may satisfactorily be approximated by a straight line. Hence let us assume $N_t(T) = a_{1t} + b_{1t} T$

and

$$\bar{W}_t(T) = a_{2t} + b_{2t} T$$

in the usual notation. Then

$$P_{4t} = b_{2t} [a_{1t} + b_{1t}/2] \tag{10}$$

Assumption V

$N_t(T) = a_{1t} + b_{1t}\, T$

and

$\bar{W}_t(T) = \bar{W}_t\, eG_t\, T$

Then

$$P_{5t} = B_{t+1} - B_t - \frac{b_{1t}\bar{W}_t}{G_t}[eG_t - 1] \quad (11)$$

$$= B_t\tau^1 - B_t - b_{1t}\bar{W}_t \quad (12)$$

When $b_{1t}\, \bar{W}_t\, G_t$ is negligible.

Assumption VI

$N_t(T) = N_t\, e^{-Z_t\, T}$

and

$\bar{W}_t(T) = a_{2t} + b_{2t}\, T$

Then

$$P_{6t} = N_t\, b_{2t}\, [1 - e^{-Z_t}] / Z_t \quad (13)$$

$$= N_t\, b_{2t} \quad (14)$$

when $N_t\, b_2\, Z_t$ is negligible. The form (13) has been considered by Gulin and Rudenko (1973).

Assumption VII

$N_t(T) = a_{1t} + b_{1t}\, T$

and no assumption be made on $\bar{W}_t$. Then

$$P_{7t} = B_{t+1} - B_t - b_{1t}\, \bar{\bar{W}}_t \quad (15)$$

where $\bar{\bar{W}}_t$ is the mean of $\bar{W}_t$ in $(t, t+1)$.

Assumption VIII

$\bar{W}_t(T) = a_{2t} + b_{2t}\, T$

and no assumption be made on N_t. Then

$$P_{8t} = b_{2t}\, \bar{N}_t \quad (16)$$

Here it may be noted that if $\bar{N}_t$ is estimated from a simple average $(N_t + N_{t+1})/2$ and $\bar{W}_t$ from $(\bar{W}_t + \bar{W}_{t+1})/2$ then P_{4t}, P_{7t} and P_{8t} are one and the same. Normally $\bar{N}_t$ is estimated from the average of N_t and N_{t+1}. Similarly $\bar{W}_t$. As such for all practical purposes P_{4t}, P_{7t} and P_{8t} are not different.

Assumption IX: von Bertalanffy's model.

Case i : Growth is isometric.

$$W_t = W_\infty\, [1 - e^{-K(t - t_0)}]^3$$

and $N_t = N_0\, e^{-Z_t}$

Then the production function is of the form (2).

Case ii : Growth is allometric.

and $W_t = W_\infty\, [1 - e^{-Kd(t - t_0)}]^{n/d}$

$N_t = N_0\, e^{-Z_t}$

where $d = n - m$ and m is the exponent obtained in the relationship of length and surface area of fish as defined by von Bertalanffy. Similarly n is the exponent derived in the relationship between length and weight of fish (Taylor, 1962). In this case

$$P_t = C \int_{x_2}^{x_1} x^{p-1}\, (1-x)^{q-1}\, dx \quad (17)$$

where t_r is the age of fish beyond which fish are not available for catch;

$C = (R'/d) n\, W_\infty\, e^{(F+M)\,(t_{\rho 1} - t_0)}$;

$x_1 = e^{-Kd(t_{\rho 1} - t_0)}$

$x_2 = e^{-Kd\,(t_\lambda - t_0)}$.

Other symbols have the same connotation as in (2). Now (17) is an incomplete beta function which can be evaluated.

No doubt, by assuming different forms for W_t and N_t with 't' many such production functions can be evaluated. However, the assumptions on which the above production functions are based, almost cover the growth forms in currency in fishery research. By empirical studies if some other functions are found to be fitting better, then on the basis of such arrived at functions of W_t and N_t production functions can be found out.

Estimation of production

To estimate production, using the above production functions, we require estimates for N_t, W_t, G_t, Z_t etc. This we shall see in this section.

For estimation of N_t vast literature is available. Seber (1973), Robson and Regier (1964) and others have dealt with this problem. From these methods any appropriate method of estimation may be chosen and N_t estimated. From the sample or subsample taken for estimation of N_t corresponding observations on weight will give an estimate for $\bar{W}_t$. However, independent samples for estimating N_t and $\bar{W}_t$ would simplify the variance estimates. Hence throughout this paper it is assumed that estimates on N_t and $\bar{W}_t$ are obtained independently so that their covariance term vanishes. Now

$$\hat{G}_t = \log_e \hat{\bar{W}}_{t+1} - \log_e \hat{\bar{W}}_t \tag{18}$$

since we are considering unit time segments. Similarly

$$\hat{Z}_t = \log_e \hat{N}_t - \log_e \hat{N}_{t+1} \tag{19}$$

$$\hat{B}_t = \hat{N}_t \hat{\bar{W}}_t \tag{20}$$

$$\hat{\bar{B}}_t = (\hat{B}_{t+1} + \hat{B}_t)/2 \text{ and } \hat{\bar{N}}_t = (\hat{N}_{t+1} + \hat{N}_t)/2$$

Using the above estimates P_{1t} to P_{3t} can be found out.

Then for P_{4t} to P_{8t} we have $\hat{a}_{1t} = \hat{N}_t$, $\hat{b}_{1t} = \hat{N}_{t+1} - \hat{N}_t$; $\hat{a}_{2t} = \hat{\bar{W}}_t$ and $\hat{b}_{2t} = \hat{\bar{W}}_{t+1} - \hat{\bar{W}}_t$. Thus all functions P_{1t} to P_{8t} can be estimated. For the rest we require estimates of F, M, K, t, t_0 etc. Beverton and Holt (1957), Paulik and Gales (1964) etc. have given methods to estimate these parameters. Since we are dealing with culture aspects these estimates are not considered in this paper.

Variance functions

Many biological functions suffer for want of corresponding variance functions. In this section let us find out variance functions for some production functions evaluated above. In doing so we shall have minimum assumptions so that variance functions evaluated on the basis of these assumptions do not differ much from their exact counterparts.

The estimate $\hat{V}(\hat{N})$ depends on the procedure by which N is obtained and $\hat{V}(\hat{N})$ is readily available from Seber (1973) and others.

Now

$$\hat{V}(\hat{\bar{W}}_t) = \hat{V}(\hat{W}_t)/r$$

and

$$\hat{V}(\hat{W}_t) = [\Sigma W^2_{ti} - (\Sigma W_{ti})^2/r] / (r-1) \tag{21}$$

From Kendall and Stuart (1963)

$$\hat{V}(\log_e \hat{\bar{W}}_t) = \hat{V}(\hat{\bar{W}}_t) / \hat{\bar{W}}^2_t \tag{22}$$

Hence

$$\hat{V}(\hat{G}_t) = \hat{V}(\log_e \hat{\overline{W}}_t) + \hat{V}(\log_e \hat{\overline{W}}_{t+1})$$

$$= \frac{\hat{V}(\hat{\overline{W}}_t)}{\hat{\overline{W}}_t^2} + \frac{\hat{V}(\hat{\overline{W}}_{t+1})}{\hat{\overline{W}}^2_{t+1}} \tag{23}$$

Thus (23) avoids the recourse to subsampling to obtain $V(G_t)$ as suggested by Chapman (1971). In (23) and also hereafter, the covariance term is omitted since $\overline{W}_t$ and $\overline{W}_{t+1}$ are independently estimated. Similarly

$$\hat{V}(\hat{Z}_t) = \frac{\hat{V}(\hat{N}_t)}{\hat{N}_t^2} + \frac{\hat{V}(\hat{N}_{t+1})}{\hat{N}_{t+1}^2} \tag{24}$$

$$\hat{V}(\hat{B}_t) = [\hat{V}(\hat{B}_t) + \hat{V}(\hat{B}_{t+1})]/4 \tag{25}$$

where

$$\hat{V}(\hat{B}_t) = \hat{V}(\hat{N}_t \hat{\overline{W}}_t) = \hat{N}_t^2 \hat{V}(\hat{\overline{W}}_t) + \hat{\overline{W}}_t^2 \hat{V}(\hat{N}_t) \tag{26}$$

etc. On the basis of the above we shall evaluate here variance functions for all P_{1t} to P_{8t} except P_{5t}. Now

$$\hat{P}_{1t} = \hat{N}_t \hat{\overline{W}}_t \hat{G}_t [{}_e(\hat{G}_t - \hat{Z}_t) - 1]/(\hat{G}_t - \hat{Z}_t)$$

$$= \hat{G}_t [\hat{N}_{t+1} \hat{\overline{W}}_{t+1} - \hat{N}_t \hat{\overline{W}}_t] / (\hat{G}_t - \hat{Z}_t)$$

$$= \hat{G}_t (\hat{B}_{t+1} - \hat{B}_t) / (\hat{G}_t - \hat{Z}_t)$$

$$= N_r / D_r \text{ (say)}$$

and

$$V(P_{1t}) = \frac{V(N_r)}{D_r^2} + \frac{[E(N_r)]^2 V(D_r)}{D_r^4} - 2E(N_r) \operatorname{Cov}(N_r ; D_r) D_r^3 \tag{27}$$

From A_1 to A_9 of the appendix

$$E(N_r) = E(\hat{B}_{t+1} \hat{G}_t) - E(\hat{B}_t \hat{G}_t)$$

$$= (B_{t+1} - B_t) \lambda G_t + \rho(\hat{\overline{W}}_t) \left(B_{t+1} \frac{V(\hat{\overline{W}}_{t+1})}{\overline{W}_{t+1}^2} + B_t \frac{V(\hat{\overline{W}}_t)}{\overline{W}_t^2} \right) \tag{28}$$

$$V(N_r) = G_t^2 [V(\hat{B}_{t+1}) + V(\hat{B}_t)] + (B_{t+1} - B_t)^2 V(\hat{G}_t)$$

$$+ 2(B_{t+1} - B_t) G_t \rho(\hat{\overline{W}}_t) \left[B_{t+1} \frac{V(\hat{\overline{W}}_{t+1})}{\overline{W}^2_{t+1}} + B_t \frac{V(\hat{\overline{W}}_t)}{\overline{W}_t^2} \right] \tag{29}$$

$$\mathrm{Cov}\,(N_r, D_r) = G_t \left\{ \rho\,(\hat{\bar{\bar{W}}}_t) \left[B_{t+1} \frac{V(\hat{\bar{\bar{W}}}_{t+1})}{\bar{\bar{W}}^2_{t+1}} + \frac{V(\hat{\bar{\bar{W}}}_t)}{\bar{\bar{W}}_t^2} B_t \right] + \right.$$

$$\left. \rho\,(\hat{N}_t) \left[B_{t+1} \frac{V(\hat{N}_{t+1})}{N^2_{t+1}} + B_t \frac{V(\hat{N}_t)}{N_t^2} \right] \right\} + (B_{t+1} - B_t) \left[\frac{V(\hat{\bar{\bar{W}}}_{t+1})}{\bar{\bar{W}}^2_{t+1}} + \frac{V(\hat{\bar{\bar{W}}}_t)}{\bar{\bar{W}}_t^2} \right]$$

$$+ \rho\,(\hat{N}_t)\, \rho\,(\hat{\bar{\bar{W}}}_t) \left[B_{t+1} \frac{V(\hat{N}_{t+1})}{N^2_{t+1}} \frac{V(\hat{\bar{\bar{W}}}_{t+1})}{\bar{\bar{W}}^2_{t+1}} - B_t \frac{V(\hat{N}_t)}{N_t^2} \frac{V(\hat{\bar{\bar{W}}}_t)}{\bar{\bar{W}}_t^2} \right] \tag{30}$$

where $\rho\,(\hat{\bar{\bar{W}}}_t)$ is the correlation between $\hat{\bar{\bar{W}}}_t$ and $\log_e \hat{\bar{\bar{W}}}_t$ or $(\log_e \hat{\bar{\bar{W}}}_t)^2$. Similarly $\rho\,(\hat{N}_t)$ is the correlation between $\hat{N}_t$ and $\log_e \hat{N}_t$ or $(\log_e \hat{N}_t)^2$. In this paper it is assumed that

$$\rho\,(\hat{\bar{\bar{W}}}_t) = \rho\,(\hat{\bar{\bar{W}}}_{t+1}) \text{ and } \rho\,(\hat{N}_t) = \rho\,(\hat{N}_{t+1}).$$

$$V(D_r) = V(\hat{G}_t) + V(\hat{Z}_t) \tag{31}$$

Using (28)—(31), $V(\hat{P}_{1t})$ can be estimated. No doubt the expression is a complex one. Similarly other functions appearing under P_{1t} can be dealt with. Now

$$\hat{P}_{2t} = \hat{G}_t \hat{\bar{B}}_t = \hat{G}_t\, (\hat{B}_t + \hat{B}_{t+1})\,/\,2.$$

Hence

$$V(\hat{P}_{2t}) = \bar{B}_t^2\, V(\hat{G}_t) + G_t^2\, V(\hat{\bar{B}}_t) + 2G_t\, \hat{\bar{B}}_t\ \mathrm{Cov}\,(\hat{G}_t, \hat{\bar{B}}_t) \tag{32}$$

From (A_3) of the appendix we have

$$\mathrm{Cov}\,(\hat{G}_t\,;\, \hat{B}_t) = -\,\rho\,(\hat{\bar{\bar{W}}}_t)\, B_t\, V(\hat{\bar{\bar{W}}}_t)\,/\,\bar{\bar{W}}_t^2 \tag{33}$$

Since $\rho\,(\hat{\bar{\bar{W}}}_t)$ is the correlation between $\hat{\bar{\bar{W}}}_t$ and $\log_e \hat{\bar{\bar{W}}}_t$, $\rho\,(\hat{\bar{\bar{W}}}_t)$ is always positive. Hence we have the result that the covariance between $\hat{G}_t$ and $\hat{B}_t$ is negative. Similarly from (A_4),

$$2\,\mathrm{Cov}\,(\hat{G}_t\,;\, \hat{\bar{B}}_t) = \rho\,(\hat{\bar{\bar{W}}}_t) \left(B_{t+1} \frac{V(\hat{\bar{\bar{W}}}_{t+1})}{\bar{\bar{W}}_{t+1}^2} - B_t \frac{V(\hat{\bar{\bar{W}}}_t)}{\bar{\bar{W}}_t^2} \right) \tag{34}$$

From (34) we cannot say with certainty about the sign of Cov $(\hat{G}_t, \hat{\bar{B}}_t)$. However, in the example considered in the end it is found (34) is negative throughout. From the above

$$V(\hat{P}_{2t}) = \bar{B}_t^2\, V(\hat{G}_t) + G_t^2\, V(\hat{\bar{B}}_t) + G_t\, \bar{B}_t\, \rho\,(\hat{\bar{\bar{W}}}_t) \left(B_{t+1} \frac{V(\hat{\bar{\bar{W}}}_{t+1})}{\bar{\bar{W}}_{t+1}^2} - B_t \frac{V(\hat{\bar{\bar{W}}}_t)}{\bar{\bar{W}}_t^2} \right) \tag{35}$$

From the example it is noticed that $\rho\,(\hat{\bar{\bar{W}}}_t)$ is almost equal to unity. As such covariance term in (35) may not be negligible as Chapman (1971) has assumed. Now $\hat{P}_{3t} = \hat{B}_{t+1} - \hat{B}_t + \hat{Z}_t\, \hat{\bar{B}}_t$.

Hence $V(\hat{P}_{3t}) = V(\hat{B}_{t+1} - \hat{B}_t) + V(\hat{Z}_t\, \hat{\bar{B}}_t) + 2\,\mathrm{Cov}\,[\,(\hat{B}_{t+1} - \hat{B}_t)\,;\, \hat{Z}_t\, \hat{\bar{B}}_t\,]$ (36)

From (A_5) of the appendix

$$V(\hat{Z}_t\,\hat{\bar{B}}_t)=Z_t^2\,V(\hat{\bar{B}}_t)+\bar{B}_t^2\,V(\hat{Z}_t)+\bar{B}_t\,Z_t\,\rho\,(\hat{N}_t)\left(B_t\frac{V(\hat{N}_t)}{N_t^2}-B_{t+1}\frac{V(\hat{N}_{t+1})}{N_{t+1}^2}\right) \tag{37}$$

and from (A_{10}) & (A_{11})

$$\begin{aligned}2\,\mathrm{Cov}\,[(\hat{B}_{t+1}-\hat{B}_t)\,;\,\hat{Z}_t\,\hat{\bar{B}}_t] = Z_t\Bigg\{ & B_{t+1}^2\frac{V(\hat{N}_{t+1})}{N_{t+1}^2}\,\frac{V(\hat{\bar{W}}_{t+1})}{\bar{W}_{t+1}^2}-B_t^2\frac{V(\hat{N}_t)}{N_t^2}\,\frac{V(\hat{\bar{W}}_t)}{\bar{W}_t^2}\\ &+V(\hat{B}_{t+1})-V(\hat{B}_t)\Bigg\}-\rho\,(\hat{N}_t)\left\{B_{t+1}^2\frac{V(\hat{N}_{t+1})}{N_{t+1}^2}\left[2\frac{V(\hat{\bar{W}}_{t+1})}{\bar{W}_{t+1}^2}+1\right]+B_t^2\frac{V(\hat{N}_t)}{N_t^2}\,.\right.\\ &\left.\left[2\frac{V(\hat{\bar{W}}_t)}{\bar{W}_t^2}+1\right]+B_{t+1}B_t\left(\frac{V(\hat{N}_{t+1})}{N_{t+1}^2}+\frac{V(\hat{N}_t)}{N_t^2}\right)\right\}\end{aligned} \tag{38}$$

Substituting (37) and (38) in (36), $V(\hat{P}_{3t})$ can be found out. This function is also a complex one. Now

$$\hat{P}_{4t}=\hat{P}_{7t}=\hat{P}_{8t}=(\hat{N}_t+\hat{N}_{t+1})\,(\hat{\bar{W}}_{t+1}-\hat{\bar{W}}_t)\;/\;2$$

Hence

$$V(\hat{P}_{4t})=\{\,(N_t+N_{t+1})^2\,[\,V(\hat{\bar{W}}_{t+1})+V(\hat{\bar{W}}_t)\,]+(\bar{W}_{t+1}-\bar{W}_t)^2\,[\,V(\hat{N}_t)+V(\hat{N}_{t+1})\,]\,\}\;/\;4 \tag{39}$$

Here all are known functions and $V(\hat{P}_{4t})$ is easily estimable. Finally

$$\hat{P}_{6t}=\hat{N}_t\,\hat{b}_{2t}\,(1-e^{Z_t})\;/\;\hat{Z}_t=(\hat{\bar{W}}_{t+1}-\hat{\bar{W}}_t)\,(\hat{N}_t-\hat{N}_{t+1})\;/\;\hat{Z}_t$$

Hence

$$\begin{aligned}V(\hat{P}_{6t})=\;&V\,[\,(\hat{\bar{W}}_{t+1}-\hat{\bar{W}}_t)\,(\hat{N}_t-\hat{N}_{t+1})\,]\;/\;Z_t^2\;+\\ &(\bar{W}_{t+1}-\bar{W}_t)^2\,(N_t-N_{t+1})^2\,V(\hat{Z}_t)\;/\;Z_t^4-\\ &2\,(\bar{W}_{t+1}-\bar{W}_t)\,(N_t-N_{t+1})\,\mathrm{Cov}\,[\,(\hat{\bar{W}}_{t+1}-\hat{\bar{W}}_t)\,(\hat{N}_t-\hat{N}_{t+1})\,;\,\hat{Z}_t\,]\;/\;Z_t^3\end{aligned}$$

Only unknown term, here, is the covariance term. From (A_{12}) of the Appendix we have

$$\mathrm{Cov}\,[\,(\hat{\bar{W}}_{t+1}-\hat{\bar{W}}_t)\,(\hat{N}_t-\hat{N}_{t+1})\,;\,\hat{Z}_t\,]=\rho\,(\hat{N}_t)\,(\bar{W}_{t+1}-\bar{W}_t)\left(\frac{V(\hat{N}_t)}{N_t}+\frac{V(\hat{N}_{t+1})}{N_{t+1}}\right) \tag{40}$$

Thus

$$\begin{aligned}V(\hat{P}_{6t})=\;&\{\,(\bar{W}_{t+1}-\bar{W}_t)^2\,[\,V(\hat{N}_t)+V(\hat{N}_{t+1})\,]+(N_t-N_{t+1})^2\,[\,V(\hat{\bar{W}}_t)+V(\hat{\bar{W}}_{t+1})\,]\,\}\;/\;Z_t^2\\ &+(\bar{W}_{t+1}-\bar{W}_t)^2\,(N_t-N_{t+1})^2\left[\frac{V(\hat{N}_t)}{N_t^2}+\frac{V(\hat{N}_{t+1})}{N_{t+1}^2}\right]\;/\;Z_t^4-\\ &2\,(\bar{W}_{t+1}-\bar{W}_t)^2\,(N_t-N_{t+1})\,\rho\,(\hat{N}_t)\left[\frac{V(\hat{N}_t)}{N_t}+\frac{V(\hat{N}_{t+1})}{N_{t+1}}\right]\;/\;Z_t^3\end{aligned} \tag{41}$$

Evaluation of variance functions for the rest of the production functions is not considered in this paper as those are more complex and intractable. It may be noted that P_{4t} (P_{7t}/P_{8t}) is comfortably easier to be evaluated and so also its variance function. Moreover its variance function does not involve any term such as $\rho\,(\hat{\bar{W}}_t)$ etc. whose estimates may not be easily available. Thus in culture practices where observations are taken at short intervals production function P_{4t} / $(P_{7t}\ P_{8t})$ is the best to be considered.

Example

Now we shall take up the data given by Chapman (1971) for our analysis. Since these data contain only numbers and average weights for every month we shall proceed as follows. Let us assume that numbers are enumerated and not estimated. Hence $V(\hat{N}_t) = O$ for all 't'. To get estimates of $V(\hat{\bar{W}}_t)$ values for weight measurements are generated from random numbers as indicated below. Now $\bar{W}_t = 1.5$ g for May. Assuming that the range of W_t is in $1.0 - 2.0$ g the decimal place is filled by the help of random numbers. For example the first one digit number noted from random number table was '2'. Hence the first value of W_t is 1.2. The second number from the random number tables was '0'. Hence the second value of W_t is 2.0. In this way twenty numbers are generated with the restriction that they add up to $20 \times 1.5 = 30.0$ by making slight adjustments. Thus for each month twenty values are generated with the restriction that they have the mean given by Chapman (1971). From these values given in Table *I*, $V(\hat{\bar{W}}_t)$ are estimated and given in Table 1. Corresponding estimates for $V(\hat{G}_t)$ are obtained and given in Table 2. The estimates of P_{2t}, P_{3t}, P_{4t} and P_{6t} are also given in Table 2. The closeness of these estimates is worth noting. Variance functions for P_{3t} and P_{4t} are also found out. In this connection it is a problem to estimate $\rho(\hat{\bar{W}}_t,\ \log_e\ \hat{\bar{W}}_t)$. To get an idea about the magnitude of $\rho(\hat{\bar{W}}_t,\ \log_e\ \hat{\bar{W}}_t)$ first of all $\rho(W_t,\ \log_e\ W_t)$ was calculated and found to be almost unity. Then taking moving average of two for W_t and log values for the average, correlation was found to be almost unity. Further, moving average of three, four and five were also tried and in all these, correlation came closely to unity. On the basis of this observation, for the present example, correlation is taken as unity and thus $V(\hat{P}_{3t})$ is estimated from (35) putting $\rho(\hat{\bar{W}}_t) = 1$; one more assumption is also made that $V(\Sigma\hat{P}_{3t}) = \Sigma\ V(\hat{P}_{3t})$ and $V(\Sigma\hat{P}_{4t}) = \Sigma V(\hat{P}_{4t})$.

Variance estimates for ΣP_{3t} and ΣP_{4t} are alone found and given in Table 2 for comparison. Calculation of other variance estimates need not be difficult though they may take considerably more time. Proper computor programming will solve this problem. This would be considered subsequently.

Among the four production functions estimated, the estimates of P_3 alone does not fall within the confidence interval of either P_2 or P_4 though monthly estimates of these production functions do not vary much from each other as noted earlier. When P_4 is compared with P_2 of Ricker (1946) and Allen (1950) and P_6 of Gulin and Rudenko (op. cit) it is clear that to estimate P_4 as well as its variance function is much easier and less time consuming. The estimates of P_4 fares well with P_2 and P_6. Hence P_4 is preferable to other production functions considered here.

TABLE 1. *Generated monthly weight figures (g) with mean, variance and population number*

Generated values		Months						
		May	June	July	August	September	October	November
1	..	1.2	2.3	2.1	3.7	4.6	6.5	6.7
2	..	2.0	2.4	2.9	3.6	4.4	6.7	6.9
3	..	1.2	1.9	2.4	3.3	4.7	6.7	6.8
4	..	1.3	1.8	2.6	3.2	4.0	6.6	6.7
5	..	1.6	1.9	2.8	3.6	4.4	6.6	7.2
6	..	1.3	1.8	2.3	3.0	4.5	6.3	7.2
7	..	1.9	1.8	2.8	3.6	4.6	6.5	6.6
8	..	1.7	2.0	2.0	3.8	4.7	6.1	6.6
9	..	1.4	2.0	2.8	3.6	4.6	6.3	7.2
10	..	1.6	2.0	2.8	3.1	4.4	6.6	7.0
11	..	1.2	2.3	2.3	3.7	4.7	6.2	7.3
12	..	1.3	1.9	2.7	3.4	4.9	6.6	6.4
13	..	1.3	2.3	2.6	3.7	4.1	6.5	6.8
14	..	1.6	1.9	2.2	3.9	4.0	6.4	6.5
15	..	1.4	2.3	2.5	3.8	4.1	6.8	6.6
16	..	1.8	1.7	2.6	3.6	4.4	6.3	7.3
17	..	1.6	1.7	2.4	3.1	4.5	6.6	7.1
18	..	2.0	2.3	2.3	3.5	4.8	6.5	7.2
19	..	1.2	1.9	2.3	3.7	4.7	7.0	7.0
20	..	1.4	1.8	2.6	3.1	4.9	6.2	6.9
$\hat{\bar{W}}$	..	1.5	2.0	2.5	3.5	4.5	6.5	6.9
$V(\hat{\bar{\bar{W}}})$	..	0.0034	0.0025	0.0032	0.0036	0.0036	0.0024	0.0038
N	..	8,000	4,500	3,500	3,000	2,500	2,000	1,900

TABLE 2. *Estimates of G, Z etc.*

Period		Estimates								
		$\hat{G}$	$\hat{Z}$	$\hat{\bar{B}}$ (kg)	$\hat{b}_{2t}$ (g)	$\hat{P}_{2t}$ (kg)	$\hat{P}_{3t}$ (kg)	$\hat{P}_{4t}$ (kg)	$\hat{P}_{6t}$ (kg)	$V(\hat{G}_i)$
May-June	..	0.29	0.58	10.5	0.5	3.0	3.1	3.1	3.0	0.00213
June-July	..	0.22	0.25	8.8	0.5	1.9	1.9	2.0	2.0	0.00113
July-August	..	0.34	0.15	9.6	1.0	3.3	3.2	3.2	3.3	0.00080
Aug.-Sept.	..	0.26	0.18	10.6	1.0	2.8	2.6	2.8	2.7	0.00047
Sept.-Oct.	..	0.37	0.22	12.1	2.0	4.5	4.5	4.5	4.5	0.00024
Oct.-Nov.	..	0.06	0.05	13.0	0.4	0.8	0.7	0.8	0.8	0.00007
				Total	..	16.3	16.0	16.4	16.3	
				$V(\hat{P})$	..	0.4573	—	0.5019	—	

APPENDIX

Let us assume $E(\hat{N}_t)=N_t$; $E(\hat{\overline{W}}_t)=\overline{W}_t$; $E(\hat{G}_t)=G_t$; $\rho(\hat{N}_t)=\rho(\hat{N}_{t+1})$ and $\rho(\hat{\overline{W}}_t)=\rho(\hat{\overline{W}}_{t+1})$.

$$E(\hat{B}_t\,\hat{G}_t)=E[\hat{N}_t\,\hat{\overline{W}}_t\ (\log_e\hat{\overline{W}}_{t+1}-\log_e\hat{\overline{W}}_t)]$$

$$=N_t\,\overline{W}_t\,E(\log_e\hat{\overline{W}}_{t+1})-N_t\,E(\hat{\overline{W}}_t\log_e\hat{\overline{W}}_t)$$

$$=N_t\,\overline{W}_t\,E(\log_e\hat{\overline{W}}_{t+1})-N_t\,[\mathrm{Cov}(\hat{\overline{W}}_t;\ \log_e\hat{\overline{W}}_t)+\overline{W}_t\,E(\log_e\hat{\overline{W}}_t)]$$

$$=N_t\,\overline{W}_t\,G_t-\rho(\hat{\overline{W}}_t)\,N_t\,V(\hat{\overline{W}}_t)\ /\overline{W}_t$$

$$=B_t\,[G_t-\rho(\hat{\overline{W}}_t)\,V(\hat{\overline{W}}_t)\ /\overline{W}_t^2] \tag{A 1}$$

Similarly

$$E(\hat{B}_{t+1}\,\hat{G}_t)=B_{t+1}\,[G_t+\rho(\hat{\overline{W}}_t)\,V(\hat{\overline{W}}_{t+1})\ /\overline{W}_{t+1}^2] \tag{A 2}$$

Hence

$$\mathrm{Cov}(\hat{B}_t,\hat{G}_t)=-B_t\,\rho(\hat{\overline{W}}_t)\,V(\hat{\overline{W}}_t)\ /\overline{W}_t^2 \tag{A 3}$$

and

$$2\,\mathrm{Cov}(\hat{\overline{B}}_t,\hat{G}_t)=\rho(\hat{\overline{W}}_t)\left[B_{t+1}\frac{V(\hat{\overline{W}}_{t+1})}{\overline{W}_{t+1}^2}-B_t\frac{V(\hat{\overline{W}}_t)}{\overline{W}_t^2}\right] \tag{A 4}$$

Similarly

$$2\,\mathrm{Cov}(\hat{\overline{B}}_t,\hat{Z}_t)=\rho(\hat{N}_t)\left[B_t\frac{V(\hat{N}_t)}{N_t^2}-B_{t+1}\frac{V(\hat{N}_{t+1})}{N_{t+1}^2}\right] \tag{A 5}$$

Now assuming $\rho[\overline{W}_t,(\log_e\hat{\overline{W}}_t)^2]=\rho(\hat{\overline{W}}_t)$

$$E[\hat{\overline{W}}_t(\log_e\hat{\overline{W}}_t)^2]=\mathrm{Cov}[\hat{\overline{W}}_t,(\log_e\hat{\overline{W}}_t)^2]+\overline{W}_t\,E(\log_e\hat{\overline{W}}_t)^2$$

$$\mathrm{Cov}[\hat{\overline{W}}_t,(\log_e\hat{\overline{W}}_t)^2]=\rho(\hat{\overline{W}}_t)\sqrt{V(\hat{\overline{W}}_t)\,V[(\log_e\overline{W}_t)^2]}$$

$$=2\,\rho(\hat{\overline{W}}_t)\,E(\log_e\hat{\overline{W}}_t)\,V(\hat{\overline{W}}_t)\,/\overline{W}_t$$

where $V[(\log_e \hat{\overline{W}}_t)^2] = 4[E(\log_e \hat{\overline{W}}_t)^2]\, V(\hat{\overline{W}}_t)\ /\overline{W}_t^2$

Hence

$$E(\hat{B}_t\,\hat{G}_t^2) = B_t\,E(\log_e \hat{\overline{W}}_{t+1})^2 + N_t\,E[\hat{\overline{W}}_t\,(\log_e \hat{\overline{W}}_t)^2] - 2N_t\,E(\log_e \hat{\overline{W}}_{t+1}) \times E(\hat{\overline{W}}_t \log_e \hat{\overline{W}}_t)$$

$$= B_t\,[V(\log_e \hat{\overline{W}}_{t+1}) + \{E(\log_e \overline{W}_{t+1})\}^2] + N_t\,[\mathrm{Cov}\,\{\hat{\overline{W}}_t, (\log_e \hat{\overline{W}}_t)^2\} + \overline{W}_t\,E(\log_e \hat{\overline{W}}_t)^2] - 2N_t\,E(\log_e \hat{\overline{W}}_{t+1})\,[\mathrm{Cov}\,(\hat{\overline{W}}_t, \log_e \hat{\overline{W}}_t) + \overline{W}_t\,E(\log_e \hat{\overline{W}}_t)]$$

$$= B_t\left[G_t^2 + \frac{V(\hat{\overline{W}}_t)}{\overline{W}_t^2} - \frac{V(\hat{\overline{W}}_{t+1})}{\overline{W}_{t+1}^2} - 2\,\rho\,(\hat{\overline{W}}_t)\,G_t\,V(\hat{\overline{W}}_t)\ /\overline{W}_t^2\right] \tag{A 6}$$

Similarly

$$E(\hat{B}_{t+1}\,\hat{G}_t^2) = B_{t+1}\left[G_t^2 + \frac{V(\hat{\overline{W}}_t)}{\overline{W}_t^2} + \frac{V(\hat{\overline{W}}_{t+1})}{\overline{W}^2_{t+1}} + 2\,\rho\,(\hat{\overline{W}}_t)\,G_t\,V(\hat{\overline{W}}_{t+1})\ /\overline{W}_{t+1}\right] \tag{A 7}$$

$$E(\hat{B}_t\,\hat{G}_t\,\hat{Z}_t) = B_t\left[\rho\,(\hat{N}_t)\,\frac{V(\hat{N}_t)}{N_t^2}\,G_t - \rho\,(\hat{N}_t)\,\rho\,(\hat{\overline{W}}_t)\,\frac{V(\hat{N}_t)}{N_t^2}\,\frac{V(\hat{\overline{W}}_t)}{\overline{W}_t^2} - \rho\,(\hat{\overline{W}}_t)\,\frac{V(\hat{\overline{W}}_t)}{\overline{W}_t^2}\,Z_t + Z_t\,G_t\right] \tag{A 8}$$

and

$$E(\hat{B}_{t+1}\,\hat{G}_t\,\hat{Z}_t) = B_{t+1}\left[\rho\,(\hat{\overline{W}}_t)\,\frac{V(\hat{\overline{W}}_{t+1})}{\overline{W}_{t+1}^2}\,Z_t - \rho\,(\hat{N}_t)\,\rho\,(\hat{\overline{W}}_t)\,\frac{V(\hat{N}_{t+1})}{N_{t+1}^2}\,\frac{V(\hat{\overline{W}}_{t+1})}{\overline{W}_{t+1}^2} - \rho\,(\hat{N}_t)\,\frac{V(\hat{N}_{t+1})}{N_{t+1}^2}\,G_t + Z_t\,G_t\right] \tag{A 9}$$

Now

$$E(\hat{B}_t^2\,\hat{Z}_t) = E(\hat{\overline{W}}_t^2)\,E(\hat{N}_t^2 \log_e \hat{N}_t) - E(\hat{\overline{W}}_t)^2\,E(\hat{N}_t^2)\,E(\log_e \hat{N}_{t+1})$$

$$= [V(\hat{\overline{W}}_t) + \overline{W}_t^2]\,\{[2\,\rho\,(\hat{N}_t)\,V(\hat{N}_t) + E(\hat{N}_t^2)\,E(\log_e \hat{N}_t)] - [V(N_t) + N_t^2]\,\log_e(\hat{N}_{t+1})\}$$

$$= [V(\hat{\overline{W}}_t) + \overline{W}_t^2]\,[2\,\rho\,(\hat{N}_t)\,V(\hat{N}_t) + \{V(\hat{N}_t) + N_t^2\}\,Z_t] \tag{A 10}$$

Similarly, assuming that

$$\rho\,(\hat{N}_t, \log_e \hat{N}_t) = \rho\,(\hat{N}_t^2, \log_e \hat{N}_t)$$

$$E\,(\hat{B}_{t+1}^2\,\hat{Z}_t) = [\,V\,(\hat{\overline{W}}_{t+1}) + \overline{W}_{t+1}^2\,]\,[\,\{\,V\,(\hat{N}_{t+1}) + N_{t+1}^2\}\,Z_t - 2\,\rho\,(\hat{N}_t)\,V\,(\hat{N}_{t+1})\,] \qquad \text{(A 11)}$$

Now

$$E\,[\,(\hat{\overline{W}}_{t+1} - \hat{\overline{W}}_t)\,(\hat{N}_t - \hat{N}_{t+1})\,\hat{Z}_t\,] = (\overline{W}_{t+1} - \overline{W}_t)\,E\,[\,(\hat{N}_t - \hat{N}_{t+1})\,\hat{Z}_t\,]$$

$$= (\overline{W}_{t+1} - \overline{W}_t)\,[\,E\,(\hat{N}_t \log_e \hat{N}_t) - N_t\,(E \log_e \hat{N}_{t+1}) - N_{t+1}\,E\,(\log_e \hat{N}_t)$$

$$+ E\,(\hat{N}_{t+1} \log_e \hat{N}_{t+1})\,]$$

$$= (\overline{W}_{t+1} - \overline{W}_t)\,\left[\,\rho\,(\hat{N}_t)\left\{\frac{V\,(\hat{N}_t)}{N_t} + \frac{V\,(\hat{N}_{t+1})}{N_{t+1}}\right\} + Z_t\,(N_t - N_{t+1})\,\right] \qquad \text{(A 12)}$$

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