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8-ORDINATE SCHEME FOR FORMULATING PERIODIC VARIATIONS

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ABSTRACT

This communication is aimed at evolving a lucid scheme of computing the wave components of periodic variations of cascade nature.

The scheme is applied to three different series and is found best suited when the diurnal, semidiurnal and quarter diurnal variations of a parameter are involved.

INTRODUCTION

THOUGH ignored at the beginning of the last century, the expression of a systematic variation of periodic character as a mathematical (semi-emperical) function representing a series of waves of multiple periods, originally presented by a French Physicist Fourier* attracted subsequently the field of applied science and the harmonic analysis has its base on the Fourier series.

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The total variations over a period of 2π are split into a series of waves—the fundamental wave (single wave) occupying the full period and the higher harmonics 2nd, 3rd, 4th, etc. occupying the same interval 2π by 2, 3, 4, ... waves respectively.

If y be a periodic variable of x, (x varying from 0° to 2π) the function is given by

$$y = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + a_4 \cos 4x + \ldots\ldots + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + \ldots\ldots$$

where $(a_1 \cos x + b_1 \sin x)$, $(a_2 \cos 2x + b_2 \sin 2x)$, $(a_3 \cos 3x + b_3 \sin 3x)$, ... constitute the fundamental, 2nd, 3rd. . . . . components (waves) respectively of the periodic function. The series may have to be extended until one is satisfied with his requirements of accuracy.

* It was ironical that the Fourier series, the most famous writing in the works of mathematics presented to the Paris Academy in 1807 was rejected for its publication for want of very rigorous mathematical proofs.

When the function is known, the coefficients are given by[2]:

$$a_0 + \frac{1}{2\pi}\int_0^{2\pi} y\,dx\,;$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} y\cos kx\,dx\,;$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} y\sin kx\,dx\,;$$

where $k = 1, 2, 3, \ldots\ldots$

An Incite into the Practical Value

Let x in degrees vary from 0 to 2π and y be its function. Let the period 2π be divided into N equal parts. As N varies from 0 to N, correspondingly x varies from 0 to 2π. The values of the corresponding ordinates can be obtained from the graph of y against x or from the table containing values of x and the respective ordinate values.

Let the number of harmonics of our interest be half of N. Then the details of the wave frequencies within the period 2π would be as follows:

Harmonic (wave having a frequency of)	Total angle	cosine term	sine term
0 (No Wave)	$0x^\circ$	$a_0 \cos(0x)$	$b_0 \sin(0x)$
1	$1x^\circ$	$a_1 \cos(1x)$	$b_1 \sin(1x)$
2	$2x^\circ$	$a_2 \cos(2x)$	$b_2 \sin(2x)$
3	$3x^\circ$	$a_3 \cos(3x)$	$b_3 \sin(3x)$
.	.	.	.
.	.	.	.
p	px°	$a_p \cos(px)$	$b_p \sin(px)$
.	.	.	.
.	.	.	.
$\frac{N}{2}$	$\frac{N}{2}x^\circ$	$a_{\frac{N}{2}} \cos\left(\frac{N}{2}x\right)$	$b_{\frac{N}{2}} \sin\left(\frac{N}{2}x\right)$

As the interval 0-360° is divided into N equal parts each part will be equal to $\left(\frac{360}{N}\right)^\circ$ and the separation points be denoted by x_r ($r = 0, 1, 2, \ldots\ldots, N-1$). The corresponding values of x_r and their corresponding ordinates may be written as in Table 1.

Table 1

r (Division No.)	0	1	2	3	......	N−2	N−1
x°_r	$\left(\frac{360^\circ}{N}\right)0$	$\left(\frac{360^\circ}{N}\right)1$	$\left(\frac{360^\circ}{N}\right)2$	$\left(\frac{360^\circ}{N}\right)3$	......	$\left(\frac{360^\circ}{N}\right)(N-2)$	$\left(\frac{360^\circ}{N}\right)(N-1)$
$(px_r)^\circ$	$p\left(\frac{360^\circ}{N}\right)0$	$p\left(\frac{360^\circ}{N}\right)1$	$p\left(\frac{360^\circ}{N}\right)2$	$p\left(\frac{360^\circ}{N}\right)3$	......	$p\left(\frac{360^\circ}{N}\right)(N-2)$	$p\left(\frac{360^\circ}{N}\right)(N-1)$
y_r	y_0	y_1	y_2	y_3	......	y_{N-2}	y_{N-1}

Note that in our choice of the N-ordinates, the ordinate at the boundary of $x_r = 360°$ is omitted as the ordinate at $x_r = 0°$ is in our choice so that the total number of ordinates are equal to the total number of intervals into which the period 2π is divided. The coefficient b_0 will vanish as $\sin(0x_r) = 0$: and $a_0 \cos(0_{xr}) = a_0$. And also when the harmonic $p = \frac{N}{2}$, $\frac{N}{2} x_r$ will become alternately 360° and 180° as we pass from one division to the other from 0 to N -1. Therefore $\sin \frac{N}{2} x_r = 0$, hence $b_{\frac{N}{2}}$ will vanish, and $a_{\frac{N}{2}}$ will become alternately positive and negative.

The coefficients are given by (Lipka, 1918; Salvadori, 1948)

$$\left.\begin{aligned} a_0 &= \frac{1}{N}\sum_{r=0}^{N-1} y_r \cos 0x_r = \frac{1}{N}\sum_{r=0}^{N-1} y_r \\ a_{\frac{N}{2}} &= \frac{1}{N}\sum_{r=0}^{N-1} y_r \cos\left(\frac{N}{2}x_r\right) \end{aligned}\right\} \text{for } 0 = p = \frac{N}{2}$$

$$\left.\begin{aligned} a_p &= \frac{2}{N}\sum_{r=0}^{N-1} y_r \cos px_r \\ b_p &= \frac{2}{N}\sum_{r=0}^{N-1} y_r \sin px_r \end{aligned}\right\} \text{for } 0 \neq p \neq \frac{N}{2}$$

From the above general formulae each coefficient may be independently determined and thus each harmonic can be calculated without calculating the preceding harmonics. It means a_p and b_p are twice the average of the values of the ordinates taken at the N points multiplied by the corresponding values of cos and sin of px respectively. For the results to be accurate, the number of intervals N into which the period 2π is divided must be high. If p is the largest harmonic expected to be present in the graph or table, N must be atleast equal to 2p and sometimes much larger than this number, says Salvadori (1948).

Cascade system of waves

The waves ride over $y = a_0$. The frequency is doubled or the wave period is halved as we step down from the 1st wave to the second, from the second wave to the fourth, from the fourth wave to the eighth and so on. Let us call the system of such variations as 'cascade system'. We may come across the cascade system of variations in oceanic, atmospheric and ionospheric oscillations such as diurnal, semidiurnal and quarter diurnal variations of tides.

Limiting to the 4th harmonic the cascade system of variations can be represented by

$$y = a_0 + a_1 \cos x + a_2 \cos 2x + a_4 \cos 4x + b_1 \sin x + b_2 \sin 2x.$$

Keeping N = 8, the approximate values of the coefficients are given by

$$8a_0 = \sum_{r=0}^{7} y_r \cos(0x_r); \quad 4a_1 = \sum_{r=0}^{7} y_r \cos(x)$$

$$4a_2 = \sum_{r=0}^{7} y_r \cos(2x_r); \quad 8a_4 = \sum_{r=0}^{7} y_r \cos(4x_r)$$

$$4b_1 = \sum_{r=0}^{7} y_r \sin(1x_r); \quad 4b_2 = \sum_{r=0}^{7} y_r \sin(2x_r)$$

The multiples (cosine and sine values) of the corresponding ordinates are presented in the Table 2. The sum of the ordinates multiplied by their respective multiples in each row gives the values of the coefficient written against each row.

TABLE 2. *Multiples of the chosen eight ordinates*

x_r	0°	45°	90°	135°	180°	225°	270°	315°
y_r	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7
$8a_0$	1	1	1	1	1	1	1	1
$4a_1$	1	0.7	0	−0.7	−1	−0.7	0	0.7
$4a_2$	1	0	−1	0	1	0	−1	0
$8a_4$	1	−1	1	−1	1	−1	1	−1
$4b_1$	0	0.7	1	0.7	0	−0.7	−1	−0.7
$4b_2$	0	1	0	−1	0	1	0	−1

By making use of the Table 2, the values of the coefficients in terms of the ordinates be re-written as follows :

$$8a_0 = y_0 + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7$$
$$4a_1 = (y'_1 - y'_3) + (y'_7 - y'_5) + (y_0 - y_4)$$
$$4a_2 = (y_0 - y_2) + (y_4 - y_6)$$
$$8a_4 = (y_0 - y_1) + (y_2 - y_3) + (y_4 - y_5) + (y_6 - y_7)$$
$$4b_1 = (y_2 - y_6) + (y'_1 - y'_5) + (y'_3 - y'_7)$$
$$4b_2 = (y_1 - y_3) + (y_5 - y_7)$$

where the primed ordinates refers to 0.7 $(= \frac{1}{\sqrt{2}})$ of its corresponding value. As computations are combersome, the following scheme is designed to determine the coefficients.

The 8-ordinate scheme

x_r	0°	45°	90°	135°	180°	225°	270°	315°
y_r	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7

Arrange the ordinates in the following computing form

	y_0	y_1		y_2	y_3
	y_4	y_5		y_6	y_7
i sum	i_0	i_1		i_2	i_3
j diff.	j_0	j_1		j_2	j_3
	i_0	i_1		j_0	j_1
	i_2	i_3		j_2	j_3
k	k_0	k_1	Sum m	m_0	m_1
l	l_0	l_1	Diff. n	n_0	n_1

The values of the coefficients of the terms of the function are as given in the Table 3. The numbers appearing in the same column are multiplied by the corresponding constants appearing in the first column of the Table and are added to give 4 or 8 times the coefficients as indicated in the Table.

TABLE 3. *Scheme of harmonic coefficients*

Multiplier						
0.5		m_0+n_0			m_0-n_0	
0.7		n_1			m_1	
1	k_0+k_1		l_0	k_0-k_1		l_1
	$8a_0$	$4a_1$	$4a_2$	$8a_4$	$4b_1$	$4b_2$

Another way of looking at the problem is to consider the eight linear equations in a's and b's obtained by substituting the eight sets of values of x and y in the equation

$$y_r = a_0 + a_1 \cos x_r + a_2 \cos 2x^r + a_4 \cos 4x_r + b_1 \sin x_r + b_2 \sin 2x_r$$

where r takes the values 0, 1, 2, ... , 7. It may be noted that there are eight equations and six unknowns. It means that the measured ordinates are greater in number than the coefficients. Under such condition the best way to obtain the coefficients would be by applying the least square technique (Lipka, 1941). However, it may be shown that the expressions for the coefficients obtained by the method of least squares have the same form as those given earlier. And as such, the scheme evolved for computing the coefficients still holds good.

Illustration A

The graph of fig. 1 presents the barometric pressure (mb) values observed at an interval of 2 hours starting from 7 O'clock in the morning over a period of 24 hours in the month of May 1960 at Waltair (Marty, 1965). Let these 24 hours complete the period 2π. Therefore, x in degrees starts from the initial time (in hours) of observations. The eight ordinates chosen at 3 hr interval (45°) are as follows :

x_r^*	x_0 0°	x_1 45°	x_2 90°	x_3 135°	x_4 180°	x_5 225°	x_6 270°	x_7 315°
y^r	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7
mb.	1006.8	1007.8	1006.7	1005.3	1005.7	1007.7	1006.8	1006.6

Proceeding by the scheme,

		1006.8	1007.8		1006.7	1005.3
		1005.7	1007.7		1006.8	1006.6
i	Sum	2012.5	2015.5		2013.5	2011.9
j	Diff.	1.1	0.1		−0.1	−1.3
		2012.5	2015.5		1.1	0.1
		2013.5	2011.9		−0.1	−1.3
k		4026.0	4027.4	Sum m	1.0	−1.2
l		−1.0	3.6	Diff. n	1.2	1.4

Therefore, the values of the coefficients, are

$a_0 = 1006.68$, $a_1 = 0.52$, $a_2 = -0.25$, $a_4 = -0.18$, $b_1 = -0.24$, $b_2 = 0.90$

Hence

$$y = 1006.68 + 0.52 \cos x - 0.25 \cos 2x - 0.18 \cos 4x - 0.24 \sin x + 0.9 \sin 2x$$

where y = barometric pressure (mb) and x in degrees is given by $x = 2\pi \frac{t}{24}$, t in hours

TABLE 4. *Coefficients of pressure variations over a day*

Multiplier						
0.5		1.0+1.2			1.0—1.2	
0.7		1.4			—1.2	
1	4026.0 +4027.4		—1.0	4026.0 —4027.4		3.6
	8053.4 $=8a_0$	2.08 $=4a_1$	—1.0 $=4a_2$	—1.4 $=8a_4$	—0.94 $=4b_1$	3.6 $=4b_2$

starting from the initial time of observations and the period of the fundamental wave is 24 hours (the solar day). The curve representing the equation is shown in Fig. 1. The three waves (harmonics) and their sum as determined only by 50 minutes ; and the lunar day may be approximated to 25 hours. The variations of tide during the period of a lunar day (25 hrs) are subjected to the 8-ordinate scheme. The tide level readings corresponding to the 24th

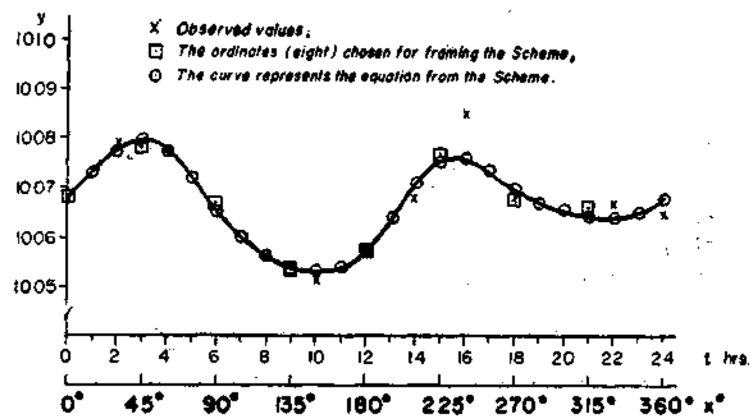

Fig. 1. The barometric pressure variations (mb) over a solar day (24 hrs).

by the 8-ordinate scheme for the chosen example of observations of variations of barometric pressure are illustrated in Fig. 2.

Illustration B

The hourly tidal observations at Cochin Port for the 29th day of February 1980 are taken from the records (Fig. 3). The lunar day (24.84 hr) and the solar day (24 hr) differ

hour on 28th day of February 1980 and that corresponding to the 1st hour on 1st March 1980 constitute respectively the values of tide at $t = 0$ and at $t = 25$ hrs. The scheme is thus,

x_r°	0	45	90	135	180	225	270	315
y_r cm	102.0	83.5	64.2	77.5	82.0	51.0	41.7	84.0

	102.0	83.5		64.2	77.5
	82.0	51.0		41.7	84.0
i Sum	184.0	134.5		105.9	161.5
j Diff.	20.0	32.5		22.5	−6.5
	184.0	134.5		20.0	32.5
	105.9	161.5		22.5	−6.5
k	289.9	296.0	Sum m	42.5	26.0
l	78.1	−27.0	Diff. n	−2.5	39.0

Therefore,

$a_0 = 73.24$, $a_1 = 11.83$, $a_2 = 19.53$
$a_4 = -0.76$, $b_1 = 10.18$, $b_2 = -6.75$

Hence the equation connecting the tide level with time of the lunar day is

$$y = 73.24 + 11.83 \cos x + 19.53 \cos 2x - 0.76 \cos 4x + 10.18 \sin x - 6.75 \sin 2x$$

where $y =$ tide level (cm) and $x = 2\pi \frac{t}{25}$ where t is in hours and 25 hr period is the lunar day which is treated as the period of the fundamental wave (first harmonic). The computed values of tidal variations in the day are plotted in Fig. 3 in the background of observed values.

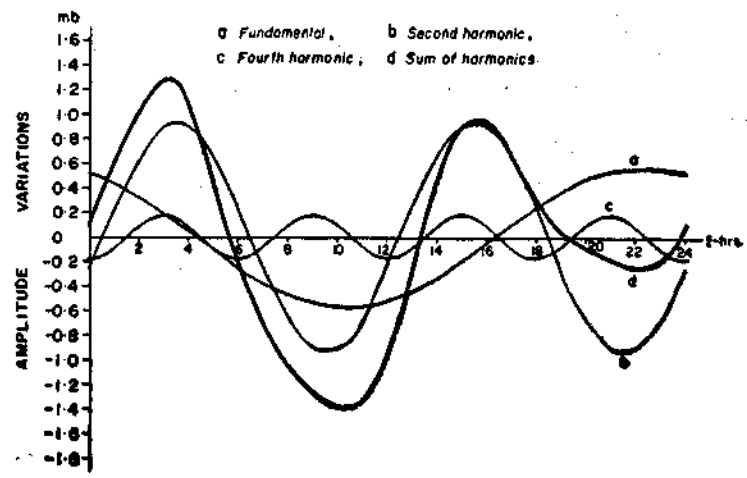

Fig. 2. The harmonic components and their sum of the barometric pressure over a_0— 1006.68 mb.

TABLE 5. *Coefficients of tidal variations over a lunar day*

Multiplier						
0.5		42.5—2.5			42.5+2.5	
0.7		39.0			26.0	
1	289.9+296.0		78.1	289.9—296.0		—27.0
	585.9 $=8a_0$	47.3 $=4a_1$	78.1 $=4a_2$	—6.1 $=8a_4$	40.7 $=4b_1$	—27.0 $=4b_2$

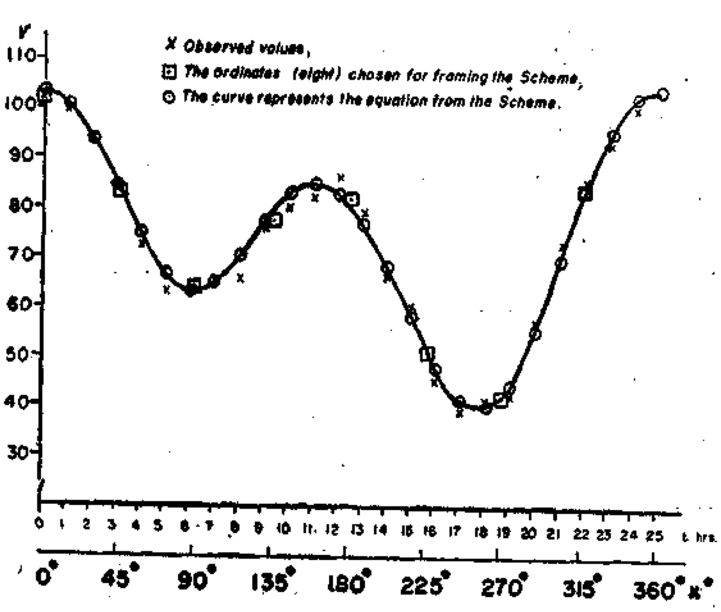

Fig. 3. The tidal variations (cm) over a lunar day (25 hrs).

Illustration C

Under the third example, the lowest low tides corresponding to each day are picked up from the hourly observations of tides at Cochin during the month of March 1980. As a lunar month refers to a complete cycle, a twenty-eight day period is chosen. In order to start the period from the very beginning of the month, the lowest low tide immediately preceding March '80 *i.e.* the last day of February '80 is considered together with its time of occurrence. This value refers to the point marked on the graph (Fig. 4) in the negative side of the time-axis.

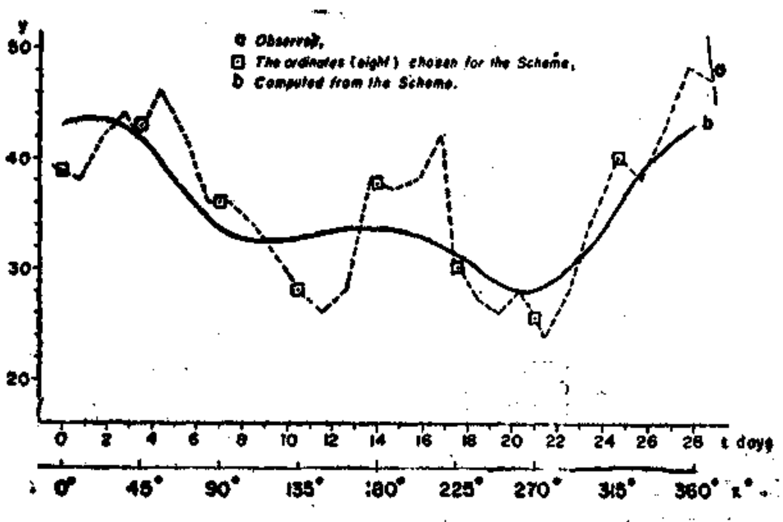

Fig. 4. The variations of the lowest low tide (cm) over a lunar month (28 days).

In this example, the fundamental period would be 28 days and x in degrees is equal to $2\pi \frac{t}{28}$ where t is in days and t = 0 refers to the mid night of 29th February/1st March 1980. It may be noted that every 3.5 days make 45° in dividing the 28 days' length of time into eight equal parts. Now,

x_r°	0	45	90	135	180	225	270	315
y_r cm.	38.7	43.0	36.0	28.0	37.7	30.0	25.5	40.0

	38.7	43.0		36.0	28.0
	37.7	30.0		25.5	40.0
i Sum	76.4	73.0		61.5	68.0
j Diff.	1.0	13.0		10.5	−12.0
	76.4	73.0		1.0	13.0
	61.5	68.0		10.5	−12.0
k	137.9	141.0	Sum m	11.5	1.0
l	14.9	5.	Diff. n	−9.5	25.0

where y = lowest low tide (cm) of the day, $x^\circ = 2\pi \frac{t}{28}$ where t is in days of the lunar month which is 28 days.

The observed and the computed values of the lowest low tide variations over the lunar month are presented in Fig. 4.

CONCLUSIONS

The 8-ordinate scheme presented here is as simple as the Runge's 6-ordinate scheme. The former incorporates the fourth harmonic instead of the third. The present scheme is handy when one has to deal with the system of variations which are expected to be 'cascade' in their resolved periodicities.

It is clear from the figures (1, 3 and 4) of the cited examples that there is a close agreement between the observed and computed sets of variations (ordinates). The nearer the system of variations to the cascade type, the closer would be the agreement between the observed and the computed total variations, as it is clear from the first two examples. In the case of the third example the influence of the solar tides during the 25-day lunar

TABLE 6. *Coefficients of tidal variations over a lunar month*

Multiplier						
0.5		11.5—9.5			11.5+9.5	
0.7		25.0			1.0	
1	137.9+141.0		14.9	137.9—141.0		5
	278.9 $=8a_0$	18.5 $=4a_1$	14.9 $=4a_2$	—3.1 $=8a_4$	11.2 $=4b_1$	5 $=4b_2$

The values of the six coefficients are

$a_0 = 34.862$; $a_1 = 4.625$; $b_1 = 2.8$
$a_2 = 3.725$; $b_2 = 1.25$; $a_4 = -0.388$

Therefore, the equation is

$$y = 34.862 + 4.625 \cos x + 2.8 \sin x + 3.725 \cos 2x + 1.25 \sin 2x - 0.388 \cos 4x.$$

month might be such as to deviate the lowest low tide variations from the cascade system. However, it is evident from Fig. 4 that the equation obtained from the scheme scans the observed values more or less through their general trend of variations during the lunar month. It is needless to say that the accuracy will improve when higher harmonics are also taken into account.

On perusal of the figures (1, 3 and 4) it may be noticed that the computed values of ordinates corresponding to the chosen (eight) ordinates need not always coincide with the latter. It means that the curve is not compelled to pass through the chosen ordinates. Nevertheless, it makes the best approximation to them, just as in the case of statistical method of least squares. This feature of the scheme is best revealed in Fig. 4. where the computed and the observed variations of the lowest low tide over a lunar month are presented in graphical form. The above mentioned smoothing feature is absent in the Runge's 6-ordinate scheme where the curve is compelled to pass through the chosen six-ordinates.

Such simplified solutions obtained from the 8-ordinate scheme may perhaps find their value in formulating simple prediction systems, especially when dealing with the diurnal variations of periodic functions.

As the harmonics are obtainable independent of each other, it can be proved that the coefficients a_3 and b_3 corresponding to the third harmonic are given by $4a_3 = 0.5\,(m_0 + n_0) + 0.7\,(-n_1)$ and $4b_3 = 0.5\,(-m_0 + n_0) + 0.7m_1$. If one is interested in the third harmonic also, two columns, one for $4a_3$ and the other for $4b_3$ can be added at suitable places in Table 3 by making use of the above equations.

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